

TOTALLY INVARIANT DIVISORS OF ENDOMORPHISMS OF PROJECTIVE SPACES

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ABSTRACT. Totally invariant divisors of endomorphisms of the projective space are expected to be always unions of linear spaces. Using logarithmic differentials we establish a lower bound for the degree of the non-normal locus of a totally invariant divisor. As a consequence we prove the linearity of totally invariant divisors for \mathbb{P}^3 .

1. INTRODUCTION

An endomorphism of a complex projective variety X is a finite morphism $f : X \rightarrow X$ of degree at least two. A totally invariant subset of f is a subvariety $D \subset X$ such that we have a set-theoretic equality $f^{-1}(D) = D$. The projective space $X = \mathbb{P}^n$ admits many endomorphisms (simply take $n+1$ homogeneous polynomials of degree m without a common zero), and it is an interesting problem to understand their dynamics [FS94]. A well-known conjecture claims that totally invariant subvarieties of endomorphisms $f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ are always linear subspaces. This conjecture¹ is known for divisors of degree $n+1$ [HN11, Thm.2.1] and smooth hypersurfaces of any degree. In fact, by results of Beauville [Bea01, Thm.], Cerveau-Lins Neto [CLN00] and Paranjape-Srinivas [PS89, Prop.8] a smooth hypersurface D of degree at least two does not admit an endomorphism, *in particular* it is not a totally invariant subset of $f : \mathbb{P}^n \rightarrow \mathbb{P}^n$. However there are examples of singular normal hypersurfaces $D \subset \mathbb{P}^n$ of degree n that admit an endomorphism $g : D \rightarrow D$ [Zha14, Ex.1.9]. One should thus ask if g is induced by an endomorphism of the projective space. The main result of this paper is a negative answer to this question:

1.1. Theorem. *Let $f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ be an endomorphism of degree at least two, and let $D \subset \mathbb{P}^n$ be a prime divisor of degree $d \geq 2$ that is totally invariant. Denote by $Z \subset D$ the non-normal locus of D . Then we have*

$$(1) \quad \deg(Z) > (d-1)^2 - \frac{n(n-1)}{2}.$$

In particular if $d \geq 1 + \sqrt{\frac{n(n-1)}{2}}$, then D is not normal.

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¹This statement is claimed in [BCS04], but the proof has a gap.

Note that if $d = n$, then the inequality (1) simplifies to

$$\deg(Z) > \frac{1}{2}(n-2)(n-1).$$

However, by a well-known result about singularities of irreducible plane curves [Fis01, 3.8], one has $\deg(Z) \leq \frac{1}{2}(n-1)(n-2)$. Thus an *irreducible* divisor D of degree n is not totally invariant. This observation significantly improves [Zha13, Thm1.1], combined with [NZ10, Thm.1.5(5) (arXiv version)] we obtain:

1.2. Corollary. *Let $f : \mathbb{P}^3 \rightarrow \mathbb{P}^3$ be an endomorphism, and let $D \subset \mathbb{P}^n$ be a prime divisor that is totally invariant. Then D is a hyperplane.*

Notation and terminology.

We work over the complex field \mathbb{C} . Let $f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ be an endomorphism, and let $D \subset \mathbb{P}^n$ be a totally invariant prime divisor. Then (e.g. by [BH14, Lemma 2.5]) there exists a unique effective divisor R such that the logarithmic ramification formula

$$K_{\mathbb{P}^n} + D = f^*(K_{\mathbb{P}^n} + D) + R$$

holds, and we call R the logarithmic ramification divisor. Since $\rho(\mathbb{P}^n) = 1$ one easily deduces that $d := \deg D \leq \deg(-K_{\mathbb{P}^n}) = n + 1$.

Given a locally free sheaf $E \rightarrow X$ over some manifold X and $x \in X$ a point, we denote by E_x the \mathbb{C} -vector space $E \otimes \mathcal{O}_X / m_x$ where $m_x \subset \mathcal{O}_X$ is the ideal sheaf of x . If $\alpha : E_1 \rightarrow E_2$ is a morphism of sheaves between locally free sheaves E_1 and E_2 , we denote by $\alpha_x : E_{1,x} \rightarrow E_{2,x}$ the linear map induced between the vector spaces.

2. THE SHEAF OF LOGARITHMIC DIFFERENTIALS

We consider the complex projective space \mathbb{P}^n of dimension $n \geq 2$.

2.1. Assumption. *In this whole section we denote by $D \subset \mathbb{P}^n$ a prime divisor of degree $d \geq 2$. We suppose that there exists a subset $W \subset \mathbb{P}^n$ of codimension at least three such that $D \setminus W$ has at most normal crossing singularities.*

2.A. Definition and Chern classes. Since D has normal crossing singularities in codimension two, the sheaf of logarithmic differentials in the sense of Saito [Sai80] and the sheaf of logarithmic differentials in the sense of Dolgachev [Dol07, Defn.2.1] coincide by [Dol07, Cor.2.2], we will denote this sheaf by $\Omega_{\mathbb{P}^n}(\log D)$. The sheaf $\Omega_{\mathbb{P}^n}(\log D)$ is reflexive (it is defined as a dual sheaf [Dol07, p.36, line -4]) and locally free in the points where D has normal crossing singularities. By [Dol07, (2.8)] there exists a residue exact sequence

$$(2) \quad 0 \rightarrow \Omega_{\mathbb{P}^n} \rightarrow \Omega_{\mathbb{P}^n}(\log D) \rightarrow \nu_*(\mathcal{O}_{\tilde{D}}) \rightarrow 0,$$

where $\nu : \tilde{D} \rightarrow D$ is the normalisation².

Our goal is to compute the first and second Chern class of the sheaf $\Omega_{\mathbb{P}^n}(\log D)$. Recall first that

$$(3) \quad c_1(\mathcal{O}_D) = D, \quad c_2(\mathcal{O}_D) = D^2.$$

Denote by $Z \subset D$ the non-normal locus of D . Since D is Cohen-Macaulay, we know by Serre's criterion that $Z \subset \mathbb{P}^n$ is empty or a projective set of pure dimension $n - 2$. We have an exact sequence

$$(4) \quad 0 \rightarrow \mathcal{O}_D \rightarrow \nu_* \mathcal{O}_{\tilde{D}} \rightarrow \mathcal{K} \rightarrow 0,$$

where \mathcal{K} is a sheaf with support on Z . Since D has normal crossings on $D \setminus W$ the restriction of (4) to $D \setminus W$ is

$$(5) \quad 0 \rightarrow \mathcal{O}_{D \setminus W} \rightarrow \nu_*(\mathcal{O}_{\tilde{D}}) \otimes \mathcal{O}_{D \setminus W} \rightarrow \mathcal{O}_{Z \cap (D \setminus W)} \rightarrow 0.$$

Since Z is empty or of pure dimension $n - 2$ and W has codimension at least three in \mathbb{P}^n , we see that W does not contain any irreducible component of Z . The second Chern class $c_2(\nu_*(\mathcal{O}_{\tilde{D}}))$ is determined by intersecting with the class of a general linear 2-dimensional subspace $P \subset \mathbb{P}^n$. Since P is disjoint from W , the sequence (5) combined with (3) yields

$$(6) \quad c_1(\nu_*(\mathcal{O}_{\tilde{D}})) = D, \quad c_2(\nu_*(\mathcal{O}_{\tilde{D}})) = D^2 - [Z].$$

Recall now that $c_1(\Omega_{\mathbb{P}^n}) = (n + 1)H$, $c_2(\Omega_{\mathbb{P}^n}) = \frac{n(n+1)}{2}H^2$ where H is the hyperplane class. Then the exact sequence (2) combined with (6) yields

$$c_2(\Omega_{\mathbb{P}^n}(\log D)) = \left(\frac{(n+1)(n-2d)}{2} + d^2 \right) H^2 - [Z].$$

Thus if we twist by $\mathcal{O}_{\mathbb{P}^n}(m)$ we obtain that

$$(7) \quad c_2(\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(m)) = \left(\frac{(n+1)(n-2d)}{2} + d^2 \right) H^2 - [Z] \\ - (n-1)(n+1-d)mH^2 + \frac{n(n-1)}{2}m^2H^2.$$

For $m = 1$ this formula simplifies to

$$(8) \quad c_2(\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1)) = (d-1)^2H^2 - [Z].$$

2.B. Global sections of $\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1)$. We now choose homogeneous coordinates X_0, \dots, X_n on \mathbb{P}^n . Since $D \subset \mathbb{P}^n$ is a prime divisor of degree $d \geq 2$, we have

$$H^0(D, \mathcal{O}_D(1)) = \langle X_0|_D, X_1|_D, \dots, X_n|_D \rangle,$$

and, for simplicity's sake, we denote by $X_0|_D, X_1|_D, \dots, X_n|_D$ also their images in $H^0(D, \nu_*(\mathcal{O}_{\tilde{D}}))$ under the natural inclusion $H^0(D, \mathcal{O}_D) \subset$

²The statement in [Dol07, (2.8)] is for a desingularisation, but since $\pi_*(\mathcal{O}_{D''}) = \mathcal{O}_{D'}$ for any birational morphism $\pi : D'' \rightarrow D'$ between normal varieties, the statement holds for the normalisation.

$H^0(D, \nu_*(\mathcal{O}_{\tilde{D}}))$. By Bott's theorem we have $H^1(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(1)) = 0$, so the cohomology sequence associated to the sequence (2) twisted by $\mathcal{O}_{\mathbb{P}^n}(1)$ shows that $X_0|_D, X_1|_D, \dots, X_n|_D$ lift to global sections of $\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1)$. In fact if we denote by f an irreducible homogeneous polynomial defining the hypersurface D , these global sections can be written in homogeneous coordinates as

$$(9) \quad \frac{d(X_0 \cdot f)}{f}, \frac{d(X_1 \cdot f)}{f}, \dots, \frac{d(X_n \cdot f)}{f}.$$

The following elementary lemma is fundamental for our proof.

2.2. Lemma. *Under the Assumption 2.1, let*

$$\alpha : \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \rightarrow \Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1)$$

be the morphism of sheaves defined by the global sections (9). Then α is surjective on $\mathbb{P}^n \setminus D_{\text{sing}}$. If $x \in D_{\text{sing}}$ is a point such that in local analytic coordinates u_1, \dots, u_n around x the hypersurface D is given by $u_1 \cdot u_2 = 0$, the linear map

$$\alpha_x : (\mathcal{O}_{\mathbb{P}^n}^{\oplus n+1})_x \rightarrow (\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1))_x$$

has rank at least $n - 1$.

For the proof recall the well-known local description of logarithmic differentials in the points where D is a normal crossings divisor: fix a point $x \in D$ and let u_1, \dots, u_n be holomorphic coordinates in an analytic neighbourhood of x . If D is given by $u_1 = 0$ in these coordinates (so $x \in D_{\text{nons}}$), then $\Omega_{\mathbb{P}^n}(\log D)$ is locally generated by

$$\frac{du_1}{u_1}, du_2, \dots, du_n.$$

If D is given by $u_1 \cdot u_2 = 0$ a set of local generators is

$$\frac{du_1}{u_1}, \frac{du_2}{u_2}, du_3, \dots, du_n.$$

Proof of the first statement. We prove the statement for $x \in D \setminus D_{\text{sing}}$, the (easier) case $x \in \mathbb{P}^n \setminus D$ is left to the reader. Up to linear coordinate change we can suppose that $x = (1 : 0 : \dots : 0)$. The affine set $U_0 := \{x \in \mathbb{P}^n \mid x_0 \neq 0\}$ is isomorphic to \mathbb{C}^n under the isomorphism

$$(X_0 : \dots : X_n) \mapsto \left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0} \right) = (Y_1, \dots, Y_n).$$

In this affine chart the forms (9) can be written as

$$(10) \quad \frac{df_b}{f_b}, \frac{Y_1 df_b}{f_b} + dY_1, \dots, \frac{Y_n df_b}{f_b} + dY_n,$$

where $f_b(Y_1, \dots, Y_n) := f(1, Y_1, \dots, Y_n)$ is the deshomogenisation of f . Since $x \in D$ is a smooth point one of the partial derivatives $\frac{\partial f_b}{\partial Y_i}(x)$ is non-zero, so

up to renumbering the coordinates Y_1, \dots, Y_n we can suppose that $\frac{\partial f_b}{\partial Y_1}(x) \neq 0$. Thus f_b, Y_2, \dots, Y_n form a set of holomorphic coordinates around x and

$$\frac{df_b}{f_b}, dY_2, \dots, dY_n$$

is a set of generators for $(\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1))|_{U_0}$ in a neighbourhood of x . Yet in the point $x = (0, \dots, 0)$ the global sections (10) are equal to $\frac{df_b}{f_b}, dY_1, \dots, dY_n$, so they contain this generating set.

Proof of the second statement. Up to linear coordinate change we can suppose that $x = (1 : 0 : \dots : 0)$ and as before we consider the affine chart $U_0 \simeq \mathbb{C}^n, Y_i = \frac{X_i}{X_0}$ and the expression (10) of the global sections in these affine coordinates. Up to renumbering we can suppose that $u_1, u_2, Y_3, \dots, Y_n$ are coordinates in an analytic neighbourhood of $(0, \dots, 0) \in \mathbb{C}^n$. Thus $(\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1))|_{U_0}$ is generated in a neighbourhood of the origin by

$$\frac{du_1}{u_1}, \frac{du_2}{u_2}, dY_3, \dots, dY_n.$$

The logarithmic forms $\frac{Y_i df_b}{f_b} + dY_i$ are equal to dY_i in the origin, so they generate the subspace

$$\langle dY_3, \dots, dY_n \rangle \subset (\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1))_x.$$

In the coordinates $u_1, u_2, Y_3, \dots, Y_n$ the polynomial f_b is equivalent to $u_1 \cdot u_2$, and

$$\frac{d(u_1 \cdot u_2)}{u_1 u_2} = \frac{du_1}{u_1} + \frac{du_2}{u_2}$$

is a non-zero element of $(\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1))_x$ which is not in the $(n-2)$ -dimensional subspace $\langle dY_3, \dots, dY_n \rangle$. Thus the global sections generate a subspace of dimension at least $n-1$. \square

3. PROOF OF THE MAIN THEOREM

The proof of Beauville's result [Bea01, Thm.] on endomorphisms of smooth hypersurfaces $D \subset \mathbb{P}^n$ is based on the fact that a global section of $\Omega_X(2)$ with isolated zeroes maps under the tangent map to a global section of $\Omega_X(2m)$ which still has isolated zeroes [ARVdV99, Lemma 1.1]. The following technical statement gives an analogue for our setting:

3.1. Lemma. *Let S be a smooth projective surface, and let E_1 be a vector bundle on S of rank $n \geq 2$. Suppose that there exists a linear subspace $V \subset H^0(S, E_1)$ such that $\dim V > \text{rk} E_1$ and the evaluation morphism*

$$ev : V \otimes \mathcal{O}_S \rightarrow E_1$$

is surjective in the complement of a finite set $Z_S \subset S$. Suppose also that for every point $x \in Z_S$ the linear map

$$ev_x : (V \otimes \mathcal{O}_S)_x \rightarrow E_{1,x}$$

has rank at least $n-1$.

Suppose that there exists a vector bundle E_2 on S of rank n and an injective morphism of sheaves

$$\varphi : E_1 \rightarrow E_2$$

such that the following holds:

- (a) The linear map $\varphi_x : E_{1,x} \rightarrow E_{2,x}$ has rank at least $n - 2$ in every point $x \in S$. The set B_S where $\text{rk}(\varphi_x) = n - 2$ is finite.
- (b) Denote by $R_S \subset S$ the closed set such that $\text{rk}(\varphi_x) < n$. Then R_S is disjoint from Z_S .

Then we have $c_2(E_1) \leq c_2(E_2)$.

Proof. Denote by $|V|$ the projective space associated to the vector space V . Consider the projective set

$$B := \{(x, \sigma) \in X \times |V| \mid \varphi(\text{ev}(\sigma(x))) = 0\},$$

and denote by $p_1 : B \rightarrow X$ and $p_2 : B \rightarrow |V|$ the natural projections. If $x \in B_S \subset R_S$, then $x \notin Z_S$ by hypothesis (b). Thus $(\varphi \circ \text{ev})_x$ has rank $n - 2$ and $\dim p_1^{-1}(x) = \dim V - n + 1$. Analogously if $x \in R_S \setminus B_S$ (resp. $x \in Z_S$), then $\dim p_1^{-1}(x) = \dim V - n$. Finally for $x \in S \setminus (R_S \cup Z_S)$ we obviously have $\dim p_1^{-1}(x) = \dim V - n - 1$. Thus we see that all the irreducible components of B have dimension at most $\dim V - n + 1$.

We will now argue by induction on the rank n .

Start of the induction: $n = 2$. Then all the irreducible components have dimension at most $\dim V - 1 = \dim |V|$, so the general fibre of p_2 is finite or empty. Hence for a general $\sigma \in |V|$, we have an induced section

$$\mathcal{O}_S \xrightarrow{\sigma} E_1 \xrightarrow{\varphi} E_2$$

of E_2 which vanishes at most in finitely many points (so it computes $c_2(E_2)$). In particular the section $\mathcal{O}_S \xrightarrow{\sigma} E_1$ vanishes at most in finitely many points and clearly $c_2(E_1) \leq c_2(E_2)$.

Induction step: $n > 2$. In this case all the irreducible components have dimension at most $\dim V - 1 < \dim |V|$, so the general p_2 -fibre is empty. Thus a general $\sigma \in |V|$ defines a morphism

$$\mathcal{O}_S \xrightarrow{\sigma} E_1 \xrightarrow{\varphi} E_2$$

that does not vanish, hence it defines a trivial subbundle of both E_2 and E_1 . In particular the quotients E_2/\mathcal{O}_S and E_1/\mathcal{O}_S are locally free and it is easy to check that the space of global sections $V/\mathbb{C}\sigma$ and the induced map $\bar{\varphi} : E_1/\mathcal{O}_S \rightarrow E_2/\mathcal{O}_S$ still satisfy the conditions of the lemma. Since $c_2(E_i) = c_2(E_i/\mathcal{O}_S)$ we can conclude. \square

Proof of Theorem 1.1. Since \mathbb{P}^n has Picard number one, the endomorphism f is polarised, i.e. we have $f^*H \equiv mH$ for some $m \in \mathbb{N}$ and H the hyperplane class. Since D is totally invariant, we know by [BH14, Cor.3.3] (cf. also [HN11, Prop.2.4]) that the pair (\mathbb{P}^n, D) is log-canonical. Since D

is Cohen-Macaulay its non-normal locus Z has pure dimension $n - 2$ and every irreducible component of Z is an lc centre of the pair (X, D) . Thus we know by [BH14, Cor.3.3] that (up to replacing f by some iterate f^l) every irreducible component of Z is totally invariant and not contained in the logarithmic ramification divisor R . Since D is totally invariant for any iterate f^l , we can suppose from now on that these properties hold for f .

Since the pair (X, D) is log-canonical there exists a subset $W \subset \mathbb{P}^n$ of codimension at least three such that $D \setminus W$ has at most normal crossing singularities. Thus we can use the logarithmic cotangent sheaf $\Omega_{\mathbb{P}^n}(\log D)$ introduced in Section 2. Since D is a totally invariant divisor, the tangent map

$$df : f^* \Omega_{\mathbb{P}^n} \rightarrow \Omega_{\mathbb{P}^n}$$

induces an injective morphism of sheaves

$$df_{\log} : f^* \Omega_{\mathbb{P}^n}(\log D) \rightarrow \Omega_{\mathbb{P}^n}(\log D).$$

Let $P \subset \mathbb{P}^n$ be a general 2-dimensional linear subspace, and $S := f^{-1}(P)$ its preimage. Then S is a smooth surface, and we claim that

$$\varphi : f^*(\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_S \rightarrow \Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(m) \otimes \mathcal{O}_S$$

satisfies the conditions of Lemma 3.1.

Proof of the claim. Consider the $n + 1$ -dimensional subspace $V \subset H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1))$ defined by the global sections (9). By Lemma 2.2 the evaluation morphism is surjective in the complement of the singular locus D_{sing} , and if $x \in Z$ is a general point, it has rank at least $n - 1$. Since P is general of dimension two, the intersection $P \cap D_{\text{sing}}$ consists only of general points of Z , so if we denote by

$$ev_S : f^*(V \otimes \mathcal{O}_{\mathbb{P}^n}) \otimes \mathcal{O}_S \rightarrow f^*(\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_S$$

the restriction of the (pull-back of the) evaluation morphism to S it is surjective in the complement of the finite set $Z_S := f^{-1}(P \cap Z)$ and has rank at least $n - 1$ in the points of Z_S . Since Z is totally invariant, the finite set Z_S is contained in $Z \cap S$. Since Z is not contained in the logarithmic ramification divisor R and P is general, the intersection $Z \cap R \cap S$ is empty. This shows that the sets $R_S := R \cap S$ and Z_S are disjoint.

Thus we are left to show that $\text{rk} \varphi_x \geq n - 2$ for every $x \in S$ and the set B_S where equality holds is finite. For the tangent map df this is well-known: if $W \subset \mathbb{P}^n$ is a variety of dimension d and $x \in W$ is a general point, the finite map $W \rightarrow f(W)$ is étale in x , in particular the tangent map df has rank at least $\dim W$ in x . This shows that the sets

$$\{x \in \mathbb{P}^n \mid \text{rk} df_x \leq n - k\}$$

have codimension at least k in \mathbb{P}^n . Since $\Omega_{\mathbb{P}^n}$ and $\Omega_{\mathbb{P}^n}(\log D)$ identify in the complement of D we are thus left to consider points of D . Yet if $x \in D_{\text{nons}}$ (resp. $x \in Z$ general) the vector space $\Omega_{\mathbb{P}^n}(\log D)_x$ contains a linear subspace that is naturally isomorphic to $\Omega_{D,x}$ (resp. $\Omega_{Z,x}$), so we can reduce to the case of the tangent map of $f|_D$ (resp. $f|_Z$). This proves the claim.

We can now finish the proof by comparing the Chern classes. Since $f^*H \equiv mH$ we have $[S] = m^{n-2}H^{n-2}$ and $f^*[Z] = m^2(\deg Z)H^2$. Thus it follows from (8) that

$$c_2(f^*(\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_S) = ((d-1)^2 - \deg Z)m^n.$$

By (7) and Lemma 3.1 this is less or equal than

$$(11) \quad c_2(\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(m) \otimes \mathcal{O}_S) = \left(\frac{(n+1)(n-2d)}{2} + d^2 - \deg Z \right) m^{n-2} \\ - (n-1)(n+1-d)m^{n-1} + \frac{n(n-1)}{2}m^n.$$

Since we can replace f by some iterate the inequality holds for all sufficiently divisible $m \in \mathbb{N}$. Thus by considering only the terms of order m^n we obtain

$$(12) \quad (d-1)^2 - \deg Z \leq \frac{n(n-1)}{2}.$$

This inequality is always strict since otherwise we obtain

$$0 \leq \left(\frac{(n+1)(n-2d)}{2} + d^2 - \deg Z \right) m^{n-2} - (n-1)(n+1-d)m^{n-1}$$

for all sufficiently divisible $m \in \mathbb{N}$. Now recall that $d \leq n+1$ and $d = n+1$ is excluded since we suppose that D is a prime divisor [HN11, Thm.2.1]. Hence we have $-(n-1)(n+1-d) < 0$ which yields a contradiction. Thus the strict form of (12) holds, this is equivalent to our statement. \square

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